Importance sampling for dynamic systems by approximate calculation of the optimal control function

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Abstract

An iterative method for estimating the failure probability for time-dependent problems has been developed. On the first iteration a simple control function has been built using a design point weighting principle for a serial reliability problem. After time discretization, two points were chosen to construct the compound deterministic control function. It is based on the time point when the first maximum of the homogenous solution has occurred and on the point at the end of the considered time interval. An importance sampling technique is used in order to estimate the failure probability functional on a set of initial values of state space variables and time. On the second iteration, the concept of optimal control function developed by Milstein can be implemented to construct a Markov control which provides better accuracy of the failure probability estimate than the simple control function. On both iterations, a concept of changing the probability measure by the Girsanov transformation is utilized. Simultaneously the CPU time is substantially reduced compared with the crude Monte Carlo procedure.

1 Introduction. Reliability and failure probability.

The evaluation of system performance is one of the objectives of mathematical modelling and numerical analysis of dynamical systems. The Monte Carlo method is universal for estimating different types of functionals defining the system reliability, in particular the probability of failure when the system response exceeds certain critical levels. However, since such failure probabilities are often very small, the standard implementation of the Monte Carlo method requires an inordinately large sample to secure acceptably accurate estimates of such quantities. Therefore, as in the time-independent reliability, the methods of variance reduction, also referred to as importance sampling techniques, can be used. In this paper we investigate the use of methods from the theory of stochastic control and the Girsanov transformation for developing a practical importance sampling method for dynamical systems. In particular, an iterative importance sampling method is presented using both concepts of deterministic and Markov control functions.

Performance and design requirements of a dynamical system restrain the acceptable values of the response to the safe domain $D_S \in \mathbf{R}^n$. The probability $p_S(T) = P\{E_S\}$, where $E_S = \{\omega : X(t) = X(t,\omega) \in D_S \text{ for every } 0 < t \leq T\}$, which is the probability that the response $X(t)$ will stay inside the safe domain throughout the time interval $(0, T]$. For simplicity it has been assumed that $X(0) = 0$. $p_S(T)$ is referred to as the *reliability* of the system. Thus the probability of failure $p_F(T)$ can be defined as $p_F(T) = 1 - p_S(T)$. The response is represented by a continuous stochastic process $X(t,\omega) : (0,T] \times \Omega \to \mathbf{R}^n$ defined on a (complete) probability space $(\Omega, \mathcal{F}, P)$ and for the time interval $(0, T]$, where Ω is a space of elementary events, $\mathcal{F}$ is a σ-algebra of measurable sets of Ω and $P : \mathcal{F} \to [0, 1]$ is a Borel probability measure (Øksendal 1998). Thus the probability of failure is given

$$p_F(T) = 1 - \int_{E_S} dP(\omega) = \int_{\Omega} I[g(X(\cdot,\omega))]\, dP(\omega) = E(I[g(X)]) \tag{1}$$

where $I[g(X)]$ is an indicator function defined as follows: $I[g(X)] = 0$ for $\omega \in E_S$, $I[g(X)] = 1$ otherwise.

The dynamic response of the system is assumed to be given by an Itô stochastic differential equation (SDE)

$$dX(t) = m(t,X)dt + \sigma(t,X)dW(t), \quad 0 \le s \le t \le T, \quad X(s) = x. \tag{2}$$

where $X(t) = (X_1(t), X_2(t), \dots, X_n(t))^T \in \mathbf{R}^n$ and $W(t) \in \mathbf{R}$ is a standard Wiener process with respect to the measure P. $m(t,X)$, $\sigma(t,X) \in \mathbf{R}^n$ are drift and diffusion coefficients, respectively, which satisfy suitable Lipschitz and growth conditions (Øksendal 1998). The mean value estimator of the failure probability by the Monte Carlo method is $\widehat{p}_f(T) = \frac{1}{N}\sum_{i=1}^{N} I[g(x^i)]$ where N is number of samples, x^i is a realization of the process X_t. The numerical error of the crude Monte Carlo procedure has the order of $1/\sqrt{N}$ (Milstein 1995). By the Girsanov theorem the Wiener process $W(t)$ in Eq. (2) may be substituted with another stochastic process $\widetilde{W}(t)$ defined as $(0 \le s \le t \le T)$

$$\widetilde{W}(t) = W(t) - \int_s^t u(\tau,\omega)d\tau \tag{3}$$

where is $u(\tau,\omega)$ is a real stochastic process which satisfies the Novikov condition, for details cf. Øksendal (1998). $\widetilde{W}(t)$ becomes a standard Wiener process on $(s,T]$ wrt a new measure P_{is}, which is defined below. Then the SDE (Eq. 2) takes on form $(0 \le s \le t \le T)$

$$d\widetilde{X}(t) = m(t,\widetilde{X})dt + \sigma(t,\widetilde{X})v(t,\widetilde{X})dt + \sigma(t,\widetilde{X})d\widetilde{W}(t), \quad \widetilde{X}(s) = x. \tag{4}$$

where the control function $u(t,\omega) = v(t,\widetilde{X}(t,\omega))$ is given below in Eq.(6).

The transformation of measures may be done according to the Radon-Nikodym theorem. By the Novikov condition, the function $(dP/dP_{is})^{-1}$ given in Eq.(5) is integrable. The Radon-Nikodym theorem then provides us with the measure $dP_{is} = (dP/dP_{is})^{-1}dP$ referred to above, and we can write

$$p_F(T;s,x) = \int_\Omega I[g(\widetilde{X})]\Big(\frac{dP}{dP_{is}}\Big)dP_{is}, \quad \Big(\frac{dP}{dP_{is}}\Big) = \exp\left(-\int_s^T v(\tau,\widetilde{X}(\tau))d\widetilde{W}(\tau) - \frac{1}{2}\int_s^T v^2(\tau,\widetilde{X}(\tau))d\tau\right) \tag{5}$$

where the reliability problem is now considered on the time interval $(s,T]$ with initial condition $X(s) = x$, and $p_F(T;s,x)$ denotes the associated failure probability.

The Monte Carlo estimate of the failure probability based on measure P_{is} is given as (Næss 1999) $\widehat{p}_F(T;s,x) = \frac{1}{N}\sum_{i=1}^{N} I[g(\widetilde{x}^i)]\,(dP/dP_{is})^i$. Invoking the theory of stochastic control (Øksendal 1998) it can be shown that the optimal control function for minimizing the functional $J = E\left[I^2[g(\widetilde{X})]\,(dP/dP_{is})^2\right]$ is (Milstein 1995)

$$v(s,x) = \frac{1}{p_F(T;s,x)}\left(\sigma(s,x)\cdot\frac{\partial p_F(T;s,x)}{\partial x}\right). \tag{6}$$

From Eq. (6) it follows that the optimal control function depends on the failure probability $p_F(T;s,x)$, which has to be known for all values of the arguments $(s,x) \in (0,T]\times\mathbf{R}^n$. But, of course, if the answer is known there is no need in controlling the system. On the other hand, if the failure probability can be calculated approximately on a suitable finite grid in $(0,T]\times\mathbf{R}^n$, then it is possible to construct a control function that may provide a more accurate estimation of the failure probability.

2 Numerical results and conclusion

The problem of a damped linear oscillator excited by the white noise is given with parameters undamped natural frequency $\omega_0 = 1$, a damping ratio $\xi = 0.05$, a standard deviation of the white noise $\gamma = \sqrt{0.3}$ and a life span $T = 50$

$$\ddot{Y}(t) + 2\xi\omega_0\dot{Y}(t) + \omega_0^2 Y(t) = \gamma N(t)\,; \quad Y(s) = x, \quad \dot{Y}(s) = \dot{x}. \tag{7}$$

Eq. (7) can be written as an SDE (cf. Eq. (2)). Let $X_1 = Y$ and $X_2 = \dot{Y}$, then in matrix form

$$dX(t) = A\,X(t)dt + \gamma b\,dW(t), \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -2\xi\omega_0 & -\omega_0^2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{8}$$

By replacing the Wiener process $W(t)$ by $\widetilde{W}(t)$ as explained in the previous section, Eq. (8) takes on the form

$$d\widetilde{X}(t) = A\,\widetilde{X}(t)dt + \gamma b\,v(t,\widetilde{X}(t))\,dt + \gamma b\,d\widetilde{W}(t). \tag{9}$$

If $v(t, X(t))$ is a deterministic open loop control, i.e. $v(t, X(t)) = v(t)$, then Eq. (7) has an explicit analytical solution $Y(t) = F(t-s, x, \dot{x}) + \gamma \int_s^t h(t-\tau)v(\tau)d\tau + \gamma \int_s^t h(t-\tau)dW(\tau)$ where $h(t)$ is an impulse response function and $F(t-s, x, \dot{x})$ is the homogenous solution for the displacement $Y(t)$.

On the first iteration, only the deterministic control based on FORM is used. The idea was developed by Au and Beck (2001). In Macke and Bucher (2003) the method is called the design point oscillation. The sampling probability density function is given as the sum of Gaussian distributions weighted at each time grid point by the coefficient $\frac{\Phi(-\beta(t_j))}{\sum_{i=1}^{m}\Phi(-\beta(t_i))}$, where $\Phi(\cdot)$ is the normal Gaussian distribution, $\beta(t_j)$ is the design point index at time t_j. Considering the failure function $g(x) = x_c - x_1$ ($x = (x_1, x_2)^T$), the simple control function and expression for the design point index are given as (Næss and Skaug 2000)

$$v_j(t) = \frac{x_c - F(t_j - s, x, \dot{x})}{\gamma \int_s^{t_j} h^2(t_j - \tau)d\tau} h(t_j - t) \quad \text{for} \quad s \le t \le t_j; \quad \beta_j = \beta(t_j) = \sqrt{\int_s^{t_j} v_j^2(t)dt}. \tag{10}$$

For the calculation, two time points $t_1 = T$ and $t_2 = t^*$ are used. The effect of the control function for the end of the considered time interval T was explored in Næss and Skaug (2000). The implementation of the second point t^* is essential due to nonzero initial values. As it was mentioned in Næss and Skaug (2000), the controlled paths tend to fail near the end of the time interval T. For some particular cases with initial values close to the threshold x_c, the paths tend to fail rather in the first part of the time interval around time t^* than at the end. Time t^* is the time corresponding to the occurrence of the first maximum of the homogeneous solution. Otherwise, forcing the samples to cross the threshold at T will end up with unreasonably low probabilities. In order to get a better estimate, it is suggested to compound these two control functions with weighting coefficients as mentioned above. The samples of the $x_1(t)$, $x_2(t)$ and $v(t)$ for the threshold $x_c = 4$ are shown in Fig. 1(a). The simulations are made for initial values on a rectangular grid of state-space variables (x_1, x_2) of size $(n_{x_1} \times n_{x_2})$, and on n_t time points. Due to lack of accuracy in the estimation of the functional by this compound importance sampling density, it is recommended not to use too fine grid mesh neither in state space, nor in time. The optimal number of grid points in the time domain was chosen as $n_t = 5$. It should be remarked that the time span $T - s$ should not be less than the natural quasi-period of the oscillation. If $T - s \le T_n$ ($T_n = 2\pi$ is the natural quasi-period) then controlling the paths towards the threshold entails very small probabilities, unless the initial values lie in the failure domain or very closed to it. Then the probability of failure, without any controls, will be a.s. 1. The functional becomes first order discontinuous in x_1 direction and not depending on x_2. Thus the control function (Eq. 6) in the safe domain takes on a form of uncertainty of the type $\frac{0}{0}$, and to avoid numerical difficulties, it is proposed to extrapolate the control function further in time linearly after last time step. This is not a rude violation in the mathematical sense because this kind of uncertainties may assume an arbitrary finite value ($\frac{0}{0} = M < \infty$). Further, the values are interpolated by Hermite polynomials in between the original time points.

On the second iteration during the numerical integration for each time point, the value of the optimal Markov control function is taken from 3D data calculated on the first iteration. The value of this approximate control function at each time step is sought explicitly as the point depending on the state space variables on the previous time step $v_i = v(x_1(t_{i-1}), x_2(t_{i-1}), t_{i-1})$. The examples of the sample path and Markov control function are showed in Fig. 1(b). After the sample path crosses a threshold, the control function is

set to zero. If this is not done, the control will keep the oscillations on the threshold level. This will give the estimate not of the first passage probability, but the probability of crossing the safe domain margin every time the path approaches it. In Fig. 1(c) the logarithm of the failure probability vs threshold is plotted. The crude Monte Carlo method (a solid line, $N = 10^6$) is compared with the first iteration values (squares $N = 200$) and the second iteration values (triangles $N = 100$). The final number of samples used in the Importance sampling simulation is about 10^4.

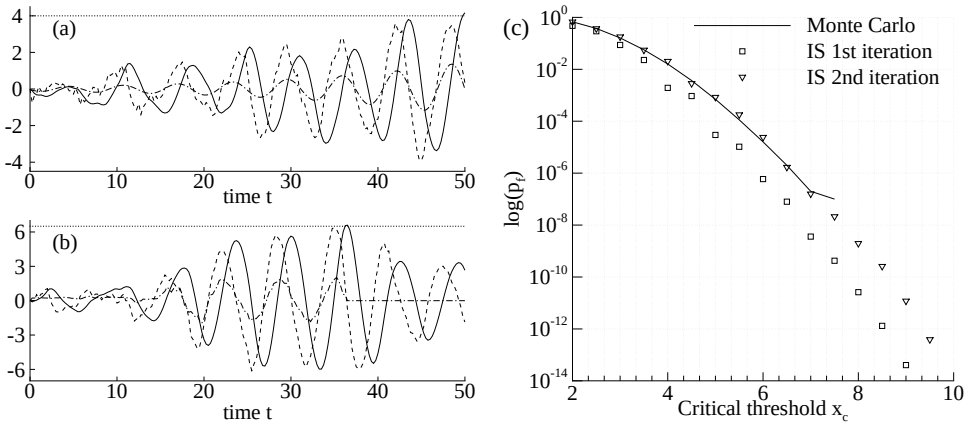

Figure 1: Examples on the sample paths (a,b (–) $x_1(t)$, (- -) $x_2(t)$, $(- \cdot -)$ $v(t)$ in (a) and $v(t, x_1, x_2)$ in (b), $(\cdots)$ threshold level, $x_c = 4$ in (a) and $x_c = 6.5$ in (b)) and failure probability estimates (c)

The second iteration estimates are very much improved compared with those obtained in the first iteration and they are in good agreement with the crude Monte Carlo estimates until this method breaks down. The advantage of the proposed method is the possibility to calculate low probabilities of order 10^{-4} and less without increasing the number of samples. The variance of the estimates and CPU time are reduced by the order of 10 compared with Monte Carlo method.

Further work is in progress to use the approximation of optimal control function for the linear oscillator for nonlinear problems (for instance, Duffing oscillator) which can be appropriately linearized.

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